

Random motion with gamma-distributed alternating velocities in biological modeling*

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Abstract

Motivated by applications in mathematical biology concerning randomly alternating motion of micro-organisms, we analyze a generalized integrated telegraph process. The random times between consecutive velocity reversals are gamma-distributed, and perform an alternating renewal process. We obtain the probability law and the mean of the process.

1 Introduction

The telegraph random process describes the motion of a particle on the real line, traveling at constant speed, whose direction is reversed at the arrival epochs of a Poisson process. After some initial works, such as [8], [11] and [17], numerous efforts have been made by numerous authors and through different methods to analyze this process. Various results on the telegraph process, including the first-passage-time density and the distribution of motion in the presence of reflecting and absorbing barriers have been obtained in [6], [7] and [19]. A wide and comprehensive review devoted to this process has recently been offered by Weiss [22], who also emphasized its relations with some physical problems.

In various applications in biomathematics the telegraph process arises as a stochastic model for systems driven by dichotomous noise (see [2], for instance). Two stochastic processes modeling the major modes of dispersal of cells or organisms in nature are introduced in [20]; under certain assumptions, the motion consisting of sequences of runs separated by reorientations with new velocities is shown to be governed by the telegraph equation. Moreover, the discrete analog of the telegraph process, i.e. the correlated random walk, is usually used as a model of the swarming behavior of myxobacteria (see [5], [9], [13] and [15]). Processes governed by hyperbolic equations are also used to describe movement

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and interaction of animals [16] and chemotaxis [10]. Moreover, the integrated telegraph process has been also used to model wear processes [3] and to describe the dynamics of the price of risky assets [4].

Many authors proposed suitable generalizations of the telegraph process, such as the 1-dimensional cases with three cyclical velocities [18], or with n values of the velocity [12], or with random velocities [21]. See also the paper by Lachal [14], where the cyclic random motion in \mathbb{R}^d with n directions is studied.

A generalized integrated telegraph process whose random times separating consecutive velocity reversals have a general distribution and perform an alternating renewal process has been studied in [1] and [23]. Along the line of such articles, in this paper we study a stochastic model for particles motion on the real line with two alternating velocities c and $-v$. The random times between consecutive reversals of direction perform an alternating renewal process and are gamma distributed, which extends the Erlang-distribution case treated in [1].

In Section 2 we introduce the stochastic process $\{(X_t, V_t); t \geq 0\}$, with X_t and V_t denoting respectively position and velocity of the particle at time t . In Section 3 we obtain a series-form of the random motion probability law for gamma-distributed random inter-renewal times, whereas the mean value of X_t conditional on initial velocity is finally obtained in Section 4.

2 The random motion

We consider a random motion on \mathbb{R} with two alternating velocities c and $-v$, with $c, v > 0$. The direction of motion is forward or backward when the velocity is c or $-v$, respectively. Velocities change according to the alternating counting process $\{N_t; t \geq 0\}$ characterized by renewal times T_1, T_2, \dots , so that T_n is the n -th random instant in which the motion changes velocity. Hence,

$$N_0 = 0, \quad N_t = \sum_{n=1}^{\infty} \mathbf{1}_{\{T_n \leq t\}}, \quad t > 0.$$

Let $\{(X_t, V_t); t \geq 0\}$ be a stochastic process on $\mathbb{R} \times \{-v, c\}$, where X_t and V_t give respectively position and velocity of the motion at time t . Assuming that $X_0 = 0$ and $v_0 \in \{-v, c\}$, for $t > 0$ we have:

$$X_t = \int_0^t V_s ds, \quad V_t = \frac{1}{2}(c - v) + \operatorname{sgn}(V_0) \frac{1}{2}(c + v)(-1)^{N_t}. \quad (1)$$

Denoting by U_k (D_k) the duration of the k -th time interval during which the motion goes forward (backward), we assume that $\{U_k; k = 1, 2, \dots\}$ and $\{D_k; k = 1, 2, \dots\}$ are mutually independent sequences of independent copies of non-negative and absolutely continuous random variables U and D .

If the motion does not change velocity in $[0, t]$, then $X_t = V_0 t$. Otherwise, if there is at least one velocity change in $[0, t]$, then $-vt < X_t < ct$ w.p. 1. Hence,

the conditional law of $\{(X_t, V_t); t \geq 0\}$ is characterized by a discrete component

$$\mathbf{P}\{X_t = yt, V_t = y \mid X_0 = 0, V_0 = y\},$$

and by an absolutely continuous component

$$p(x, t \mid y) = f(x, t \mid y) + b(x, t \mid y), \quad (2)$$

where

$$f(x, t \mid y) = \frac{\partial}{\partial x} \mathbf{P}\{X_t \leq x, V_t = c \mid X_0 = 0, V_0 = y\},$$

$$b(x, t \mid y) = \frac{\partial}{\partial x} \mathbf{P}\{X_t \leq x, V_t = -v \mid X_0 = 0, V_0 = y\},$$

with $t > 0$, $-vt < x < ct$ and $y \in \{-v, c\}$.

The formal conditional law of $\{(X_t, V_t); t \geq 0\}$ has been given in Theorem 2.1 of [1] for $V_0 = c$. Case $V_0 = -v$ can be treated by symmetry.

Explicit results for the probability law have been obtained in Theorem 3.1 of [1] when the random times U and D separating consecutive velocity reversals have Erlang distribution. This case describes the random motion of particles subject to collisions arriving according to a Poisson process with rate λ if the motion is forward and rate μ it is backward. When the motion has initial velocity c ($-v$), then the first $n-1$ ($r-1$) collisions have no effect, whereas the n th (r th) collision causes a velocity reversal. In the following section we shall treat the more general case in which the random inter-renewal times are gamma distributed.

3 Gamma-distributed random times

We assume that the random times U and D are gamma distributed with parameters (λ, α) and (μ, β) , respectively, where $\lambda, \mu > 0$ and $\alpha, \beta > 0$. Hereafter we obtain the probability law of $\{(X_t, V_t); t \geq 0\}$ for this case.

Theorem 1 *If U and D are gamma-distributed with parameters (λ, α) and (μ, β) , respectively, for $t > 0$ it is*

$$\mathbf{P}\{X_t = ct, V_t = c \mid X_0 = 0, V_0 = c\} = \frac{\Gamma(\alpha, \lambda t)}{\Gamma(\alpha)}, \quad (3)$$

and, for $-vt < x < ct$,

$$f(x, t \mid c) = \frac{1}{c+v} \left\{ e^{-\mu \bar{x}} \sum_{k=1}^{+\infty} \frac{\mu^{k\beta} (\bar{x})^{k\beta-1}}{\Gamma(k\beta)} \left[P(k\alpha, \lambda x^*) - P(k\alpha + \alpha, \lambda x^*) \right] \right\}, \quad (4)$$

$$b(x, t \mid c) = \frac{1}{c+v} \left\{ \frac{\lambda^\alpha e^{-\lambda x^*} (x^*)^{\alpha-1}}{\Gamma(\alpha)} \frac{\Gamma(\beta, \mu \bar{x})}{\Gamma(\beta)} \right. \\ \left. + e^{-\lambda x^*} \sum_{k=1}^{+\infty} \frac{\lambda^{(k+1)\alpha} (x^*)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} \left[P(k\beta, \mu \bar{x}) - P(k\beta + \beta, \mu \bar{x}) \right] \right\}, \quad (5)$$

where

$$\bar{x} = \bar{x}(x, t) = \frac{ct - x}{c + v}, \quad x^* = x^*(x, t) = \frac{vt + x}{c + v},$$

and

$$\Gamma(a, u) = \int_u^\infty t^{a-1} e^{-t} dt, \quad P(a, u) = \frac{1}{\Gamma(a)} \int_0^u t^{a-1} e^{-t} dt, \quad a > 0. \quad (6)$$

Proof. Making use of (2.4) of [1] and noting that for $k \geq 1$ the pdfs of $U^{(k)} = U_1 + \dots + U_k$ e $D^{(k)} = D_1 + \dots + D_k$ are given by

$$f_U^{(k)}(x) = \frac{\lambda^{k\alpha} x^{k\alpha-1} e^{-\lambda x}}{\Gamma(k\alpha)}, \quad f_D^{(k)}(x) = \frac{\mu^{k\beta} x^{k\beta-1} e^{-\mu x}}{\Gamma(k\beta)}, \quad x > 0, \quad (7)$$

we have

$$f(x, t | c) = \frac{1}{c + v} e^{-\mu \bar{x}} e^{\lambda \bar{x}} \sum_{k=1}^{+\infty} \frac{\mu^{k\beta} (\bar{x})^{k\beta-1}}{\Gamma(k\beta)} \frac{\lambda^{k\alpha}}{\Gamma(k\alpha)\Gamma(\alpha)} \mathcal{I}_k, \quad (8)$$

where

$$\mathcal{I}_k := \int_{\bar{x}}^t e^{-\lambda s} (s - \bar{x})^{k\alpha-1} \Gamma(\alpha, \lambda(t-s)) ds, \quad k \geq 1. \quad (9)$$

Noting that, due to (6), $\Gamma(a, u) = \Gamma(a) [1 - P(a, u)]$ we obtain

$$\mathcal{I}_k = \mathcal{I}_{1,k} - \mathcal{I}_{2,k}, \quad (10)$$

where, for $k \geq 1$

$$\mathcal{I}_{1,k} := \Gamma(\alpha) \int_{\bar{x}}^t e^{-\lambda s} (s - \bar{x})^{k\alpha-1} ds = \Gamma(k\alpha) \Gamma(\alpha) e^{-\lambda \bar{x}} \lambda^{-k\alpha} P(k\alpha, \lambda x^*), \quad (11)$$

$$\begin{aligned} \mathcal{I}_{2,k} &:= \Gamma(\alpha) \int_{\bar{x}}^t e^{-\lambda s} (s - \bar{x})^{k\alpha-1} P(\alpha, \lambda(t-s)) ds \\ &= e^{-\lambda \bar{x}} \Gamma(\alpha) \int_0^{x^*} e^{-\lambda y} y^{k\alpha-1} P(\alpha, \lambda(x^* - y)) dy = e^{-\lambda \bar{x}} \lambda^{-k\alpha} G(\lambda x^*), \end{aligned} \quad (12)$$

with

$$G(\lambda x^*) := \int_0^{\lambda x^*} e^{-u} u^{k\alpha-1} \left(\int_0^{\lambda x^*-u} e^{-\tau} \tau^{\alpha-1} d\tau \right) du. \quad (13)$$

Making use of the Laplace transform of (13) it follows

$$\mathcal{L}_z\{G(\lambda x^*)\} = \mathcal{L}_z\left\{e^{-\lambda x^*} (\lambda x^*)^{k\alpha-1}\right\} \mathcal{L}_z\left\{\int_0^{\lambda x^*} e^{-\tau} \tau^{\alpha-1} d\tau\right\} = \frac{\Gamma(k\alpha) \Gamma(\alpha)}{z(z+1)^{k\alpha+\alpha}}.$$

Hence, from identity

$$\mathcal{L}_z \{P(k\alpha + \alpha, \lambda x^*)\} = \mathcal{L}_z \left\{ \int_0^{\lambda x^*} \frac{u^{k\alpha + \alpha - 1} e^{-u}}{\Gamma(k\alpha + \alpha)} du \right\} = \frac{1}{z(z+1)^{k\alpha + \alpha}},$$

we have

$$G(\lambda x^*) = \Gamma(k\alpha) \Gamma(\alpha) P(k\alpha + \alpha, \lambda x^*).$$

Eqs. (10)÷(13) thus give

$$\mathcal{I}_k = \Gamma(k\alpha) \Gamma(\alpha) e^{-\lambda \bar{x}} \lambda^{-k\alpha} [P(k\alpha, \lambda x^*) - P(k\alpha + \alpha, \lambda x^*)]. \quad (14)$$

Eq. (4) then follows from (8) and (13). In order to obtain $b(x, t | c)$, we recall (2.5) of [1] and make use of (7) to obtain

$$\begin{aligned} b(x, t | c) &= \frac{1}{c+v} \left\{ \frac{\lambda^\alpha e^{-\lambda x^*} (x^*)^{\alpha-1}}{\Gamma(\alpha)} \frac{\Gamma(\beta, \mu \bar{x})}{\Gamma(\beta)} \right. \\ &+ e^{-\lambda x^*} e^{\mu x^*} \sum_{k=1}^{+\infty} \frac{\lambda^{(k+1)\alpha} (x^*)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} \\ &\times \left. \frac{\mu^{k\beta}}{\Gamma(\beta)\Gamma(k\beta)} \int_{x^*}^t e^{-\mu s} (s - x^*)^{k\beta-1} \Gamma(\beta, \mu(t-s)) ds \right\}. \quad (15) \end{aligned}$$

Due to (9), the integral in (15) can be calculated from (14) by interchanging x^* , β , μ with \bar{x} , α , λ , respectively. Eq. (5) then follows after some calculations. \square

Figure 1 shows density $p(x, t | c)$ as x varies in $(-vt, ct)$ for various choices of t , α and β . Hereafter we analyze the obtain the limits of densities $f(x, t | c)$ and $b(x, t | c)$ at the extreme points of interval $(-vt, ct)$, for any fixed t .

Proposition 1 *Under the assumptions of Theorem 1 we have*

$$\begin{aligned} \lim_{x \downarrow -vt} f(x, t | c) = 0, \quad \lim_{x \uparrow ct} f(x, t | c) &= \begin{cases} +\infty, & 0 < \beta < 1 \\ \frac{\mu}{c+v} [P(\alpha, \lambda t) - P(2\alpha, \lambda t)], & \beta = 1 \\ 0, & \beta > 1, \end{cases} \\ \lim_{x \uparrow ct} b(x, t | c) = \frac{\lambda^\alpha e^{-\lambda t} t^{\alpha-1}}{(c+v)\Gamma(\alpha)}, \quad \lim_{x \downarrow -vt} b(x, t | c) &= \begin{cases} +\infty, & 0 < \alpha < 1 \\ \frac{\lambda \Gamma(\beta, \mu t)}{(c+v)\Gamma(\alpha)\Gamma(\beta)}, & \alpha = 1 \\ 0, & \alpha > 1. \end{cases} \end{aligned}$$

From Proposition 1 we note that if $\alpha < 1$ ($\beta < 1$), i.e. the gamma inter-renewal density has a decreasing hazard rate, then the backward (forward) density is divergent when x approaches $-vt$ (ct). This is very different from the behavior exhibited in the case of Erlang-distributed inter-renewals (see Corollary 3.1 of [1]), when the limits are finite.

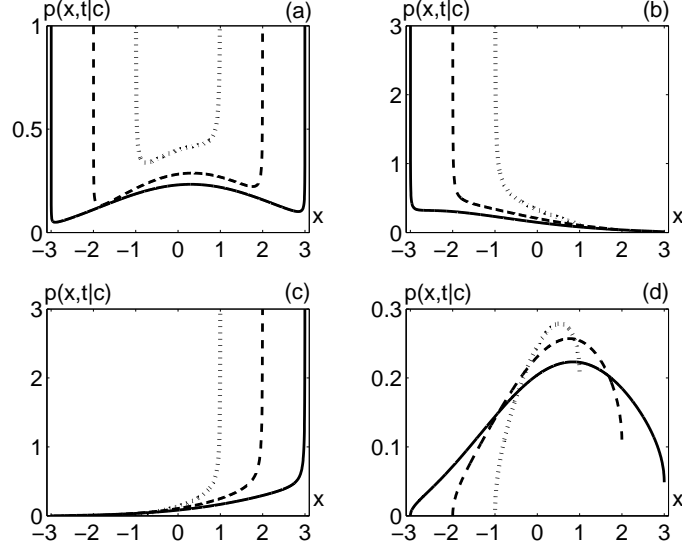


Figure 1: Density (2) for $c = v = 1$, $\lambda = \mu = 1$, and $t = 1$ (dotted line), $t = 2$ (dashed line), and $t = 3$ (solid line), with (a) $\alpha = \beta = 0.5$, (b) $\alpha = 0.5$ and $\beta = 1.5$, (c) $\alpha = 1.5$ and $\beta = 0.5$, (d) $\alpha = \beta = 1.5$.

4 Mean value

In this Section we obtain the mean value of X_t when random times U and D are identically gamma distributed.

Theorem 2 *Let U and D have gamma distribution with parameters (λ, α) . For any fixed $t \in (0, +\infty)$, we have*

$$E[X_t | V_0] = V_0 t + \frac{c+v}{\lambda} \operatorname{sgn}(V_0) \sum_{k=1}^{+\infty} (-1)^k \left[\lambda t P(k\alpha, \lambda t) - k\alpha P(k\alpha + 1, \lambda t) \right]. \quad (16)$$

Proof. Due to Eqs. (1) and recalling that $\mathbf{P}(T_k \leq s) = P(k\alpha, \lambda s)$, $s \geq 0$, it is

$$\begin{aligned} E[X_t | V_0] &= \frac{1}{2}(c-v)t + \frac{1}{2}(c+v) \operatorname{sgn}(V_0) \int_0^t E[(-1)^{N_s}] ds \\ &= \frac{1}{2}(c-v)t + \frac{1}{2}(c+v) \operatorname{sgn}(V_0) \int_0^t \left\{ 1 + 2 \sum_{k=1}^{+\infty} (-1)^k P(k\alpha, \lambda s) \right\} ds \\ &= V_0 t + (c+v) \operatorname{sgn}(V_0) \sum_{k=1}^{+\infty} (-1)^k \int_0^t P(k\alpha, \lambda s) ds. \end{aligned} \quad (17)$$

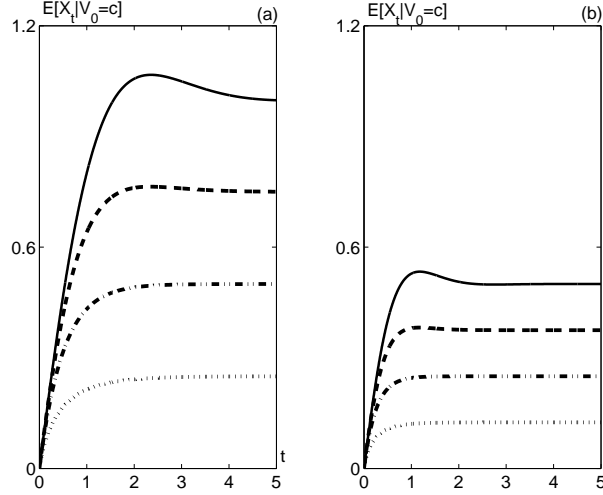


Figure 2: Mean value $E[X_t | V_0 = c]$, for $c = v = 1$ and $\alpha = 0.5$ (dotted line), $\alpha = 1$ (dash-dot line), $\alpha = 1.5$ (dash line), $\alpha = 2$ (solid line), with (a) $\lambda = 1$ and (b) $\lambda = 2$.

(Note that the above series is uniformly convergent.) Moreover, recalling (6) it is not hard to see that

$$\int_0^t P(k\alpha, \lambda s) ds = t P(k\alpha, \lambda t) - \frac{k\alpha}{\lambda} P(k\alpha + 1, \lambda t).$$

Eq. (16) then immediately follows. \square

The graphs given in Figure 2 show the mean value of X_t conditional on $V_0 = c$ for some choice of the involved parameters. We note that, being $P(\alpha, t) \sim t^{\alpha-1}/\Gamma(\alpha)$ as $t \rightarrow 0$, under the assumptions of Theorem 2 from (16) we have

$$E[X_t | V_0] \sim V_0 t \quad \text{as } t \rightarrow 0.$$

We remark that when $\alpha = n$ is integer, i.e. the random times U and D are Erlang-distributed with parameters (λ, n) , then $E[X_t | V_0]$ can be computed making use of (17) and noting that

$$E[(-1)^{N_s}] = 1 - 2e^{-\lambda s} \sum_{k=0}^{+\infty} \sum_{j=2nk+n}^{2nk+2n-1} \frac{(\lambda s)^j}{j!}.$$

For instance, in this case the following expressions hold for $t > 0$:

n	$E[X_t V_0]$
1	$\frac{(c-v)t}{2} + \frac{(c+v)}{4\lambda} \operatorname{sgn}(V_0) [1 - e^{-2\lambda t}]$
2	$\frac{(c-v)t}{2} + \frac{(c+v)}{2\lambda} \operatorname{sgn}(V_0) [1 - e^{-\lambda t} \cos(\lambda t)]$
3	$\frac{(c-v)t}{2} + \frac{(c+v)}{2\lambda} \operatorname{sgn}(V_0) \left\{ \frac{1 - e^{-2\lambda t}}{6} + \frac{4}{3} [1 - e^{-\frac{\lambda t}{2}} \cos(\frac{\sqrt{3}}{2} \lambda t)] \right\}$
4	$\frac{(c-v)t}{2} + \frac{(c+v)}{2\lambda} \operatorname{sgn}(V_0) \left\{ [1 - (1 + \frac{\sqrt{2}}{2}) e^{-\lambda t(1 - \frac{\sqrt{2}}{2})} \cos(\frac{\sqrt{2}}{2} \lambda t)] \right. \\ \left. + [1 - (1 - \frac{\sqrt{2}}{2}) e^{-\lambda t(1 + \frac{\sqrt{2}}{2})} \cos(\frac{\sqrt{2}}{2} \lambda t)] \right\}$

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